Basics of MHD

{N.B.: Read
Kulsrud, Chapt. 3, 4}

→ MHD Equations → Eulerian Fluid {1 Fluid, Large scale, slow}

①
$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{v}) = 0$$
(Continuity)

→ Lorentz, $\underline{E}$?

②
$$\rho\left(\frac{\partial v}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v}\right) = -\underline{\nabla}P + \frac{\underline{J}\times\underline{B}}{c} + \underline{f}_{body}$$
(momentum balance)

[frequently $\underline{f}_{body} = \rho\,\underline{g}$]

③
$$\frac{d}{dt}S = \frac{\partial S}{\partial t} + \underline{v}\cdot\underline{\nabla}S = 0$$
{eqn. of state more general
(isentropic fluid)

$$S = C_v \ln(P/\rho^{\gamma})$$
↑ entropy

→ [frequent form of equation of state]

④
$$\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = \left(\eta\underline{J},\ \frac{\underline{J}}{\sigma}\right)$$
(Ohms Law)

[resistivity η is usually most significant dissipation]

→ Ideal MHD

$$\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = 0$$

and ⑤ $\underline{\nabla}\cdot\underline{B}=0$

⑥ $\underline{\nabla}\times\underline{E}=-\frac{1}{c}\frac{\partial\underline{B}}{\partial t}$

⑦ $\underline{\nabla}\times\underline{B}=\frac{4\pi}{c}\underline{J}$

$\Big\}$ from Maxwell's Eqns. neglecting displacement current

→ Meaning, Restriction, Validity

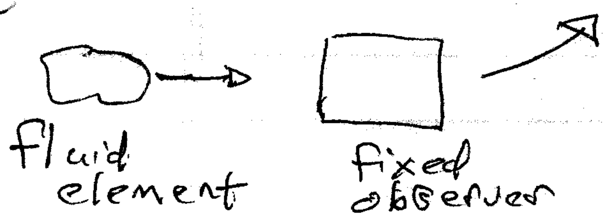
- MHD is simplest, closed, self-consistent plasma model, and the most heavily exploited for dynamical modelling.

- variants : Reduced MHD → strong $\underline{B_0}$ (tokamaks)
  - 2D MHD
  - E MHD → stationary ions (ICF)
  - FLR MHD
  - Reduced Braginskii
  - FLR MHD, Reduced Braginskii $\Big\}$ → MHD + additional effects (MFE, space)
  - hybrid → $\Big\{$ bulk - MHD; hot species - kinetic (i.e. α's energetics)
  - ⋮

glue $\Big\{$ collisions — λ_{mfp}; $B_0 \to \rho$; inertia → $\omega > k_\parallel v_{th}$

- MHD - Eulerian

"fluid element" → "glue"



here "glue" → collisions

applies $L > \lambda_{mfp}$

i → mtm
e → ci,

→ MHD is:
- 1 fluid - electrons and ions
- strongly collisional, isotropic
- low frequency
- large scale

i.e. frequencies relevant:

$$\omega \ll \Omega_{e,i};\ \omega_{p_{e,i}};\ \nu_{\substack{ee\\ ei}};\ \nu_{ii};\ \omega_{t_{e,i}}$$

etc.

scales relevant:

$$L \gg \lambda_{D_{e,i}},\ \rho_{e,i},\ c/\omega_{p_{e,i}},\ l_{mfp_{e,i}}$$

- $l_{mfp} < L$

and

collisions isotropise, equilibrate $\underline{\underline{P}}$.

$$\left(\text{i.e.}\quad \underline{\underline{P}} \sim \int d^3v\ \tilde{v}_i \tilde{v}_j\ f(\underline{x}, \underline{v}, t) \right)$$

→ Some Specific Points:

- re: continuity ①;

$$\rho = m_i n_i + m_e n_e \qquad \left\{ \begin{array}{l} \text{total density} \\ \text{ion dominated} \end{array} \right.$$

i.e. (ions control fluid inertia)

- re: momentum balance ② ;

→ $\underline{V} = \left(\int d^3v_i\, m_i \underline{v}_i f_i + \int d^3v_e\, m_e \underline{v}_e f_e\right) / \rho$

i.e. (ions control flow - $\rho \frac{d\underline{v}}{dt}$)

→ where has $\underline{E}$ gone? → $L \gg \lambda_D$ → quasi-neutrality

$$\rho_i \frac{d\underline{v}_i}{dt} = \underset{\updownarrow}{n_i Z_i \underline{E}} + n_i Z_i \frac{\underline{v}_i \times \underline{B}}{c} + \cdots$$

$$\rho_e \frac{d\underline{v}_e}{dt} = \underset{\downarrow}{-n_e Z_i \underline{E}} - \underset{\downarrow}{n_e Z_i \frac{\underline{v}_e \times \underline{B}}{c}} + \cdots$$

if add: → 0 cancel! → $\frac{\underline{J} \times \underline{B}}{c}$

$n_i = n_e$

scale

(quasi-neutrality) (Lorentz force term in momentum balance)

note also: $\rho_i, \rho_e \to \rho$

→ re-writing the $\underline{J} \times \underline{B}$ force:

$$\frac{\underline{J} \times \underline{B}}{c} = \frac{(\nabla \times \underline{B}) \times \underline{B}}{4\pi} = -\nabla\left(\frac{B^2}{8\pi}\right) + \frac{\underline{B} \cdot \nabla \underline{B}}{4\pi}$$

So can write:

$$\rho \frac{d\underline{v}}{dt} = -\underline{\nabla}\underset{\substack{\uparrow \\ \text{magnetic} \\ \text{pressure} \\ \text{(field energy density)}}}{\left(P + \frac{B^2}{8\pi}\right)} + \underset{\substack{\uparrow \\ \text{magnetic} \\ \text{tension}}}{\frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi}}$$

a) What / Why "Magnetic Tension"?

$$\underline{B} = B\hat{b} \qquad B = |\underline{B}|, \quad \hat{b} = \underline{B}/B$$

$$\underline{B}\cdot\underline{\nabla}\,\underline{B} = B\hat{b}\cdot\underline{\nabla}(B\hat{b})$$

$$= B^2 \underset{①}{\hat{b}}\cdot\underline{\nabla}\hat{b} + \underline{\hat{b}}\,\underset{②}{\underline{\hat{b}}\cdot\underline{\nabla}}\left(\frac{B^2}{2}\right)$$

$\underline{\hat{b}}\cdot\underline{\nabla}\,\underline{\hat{b}}$ → curvature of $\hat{b}$
(i.e. rate of change of $\hat{b}$ *along* itself)
$\equiv d\hat{b}/ds$

n.b. in general: curve: $\underline{x}(t)$

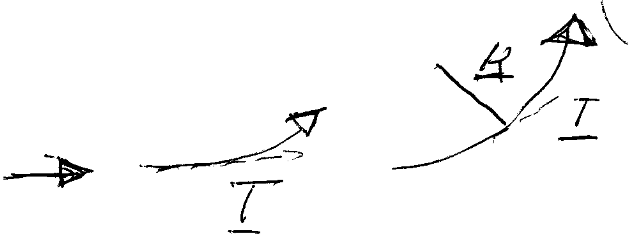
tangent: $\underline{T} = d\underline{x}/ds$

$(ds^2 = d\underline{x}\cdot d\underline{x})$ $\qquad s \equiv$ distance along curve

Curvature $\uparrow$ $\underline{\kappa} = \frac{d\underline{T}}{ds} = \frac{d\underline{T}/dt}{ds/dt} = \frac{\dot{\underline{T}}}{|\underline{V}|}$

Now: $\underline{\kappa} = \hat{b}\cdot\nabla\hat{b}$ → points (in direction of turning of $\hat{b}$), orthogonal to $\hat{b}$ tangent

i.e.



$\underline{\kappa} = d\underline{T}/ds$

∴ $\underline{\kappa} = +\frac{\hat{n}}{R_c}$ $R_c \equiv$ radius of curvature

as curved field line suggests "tension" → "magnetic tension".

b) What about ② ? → HW #1

But $\underline{J}\times\underline{B} \perp \underline{B}$ yet $\nabla\left(\frac{B^2}{8\pi}\right)$ can have component along $\underline{B}$??

→ recombining total $\underline{J}\times\underline{B}$ gives:

$$- \underline{\nabla}\left(\frac{B^2}{8\pi}\right) + B^2 \frac{\hat{b}\cdot\underline{\nabla}\hat{b}}{4\pi} + \hat{b}\,\hat{b}\cdot\underline{\nabla}\left(\frac{B^2}{8\pi}\right)$$

$$= - \nabla_\perp\left(\frac{B^2}{8\pi}\right) \cancel{- \hat{b}\hat{b}\cdot\nabla\left(\frac{B^2}{8\pi}\right)} \cancel{+ \hat{b}\,\hat{b}\cdot\underline{\nabla}\left(\frac{B^2}{8\pi}\right)} + B^2 \frac{\hat{b}\cdot\nabla\hat{b}}{4\pi}$$

$$\Rightarrow \boxed{\frac{\underline{J}\times\underline{B}}{c} = - \underline{\nabla}_\perp\left(\frac{B^2}{8\pi}\right) + B^2 \frac{\hat{b}\cdot\underline{\nabla}\hat{b}}{4\pi}}$$

3) $dE = đQ - p\,dV$ (Thermo)

$$C_v\,dT = T\,dS - p\,dV \qquad \left\{\begin{array}{l} đQ = T\,dS \\ dE = C_v\,dT \\ \text{(normalized)} \end{array}\right.$$

$$V = 1/\rho \qquad dV = -d\rho/\rho^2$$

$$p = \rho T$$

$$C_v \frac{dT}{T} = dS + \frac{d\rho}{\rho}$$

$$\Rightarrow \ln T = \frac{S}{C_v} + \ln \rho^{1/C_v}$$

$$\therefore \boxed{S' = C_v \ln\left(T/\rho^{1/C_v}\right)}$$

$$p = \rho T$$

$$\Rightarrow \quad S = c_v \ln\left(P/\rho^{(c_v+1)/c_v}\right)$$

$$= c_v \ln\left(P/\rho^{\gamma}\right)$$

$\gamma = 5/3$, ideal gas

$\left(c_v = 3/2 \text{ (normalized)}\right)$

$$\frac{dS}{dt} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(P/\rho^{\gamma}\right) = 0$$

i.e.

$$\boxed{\frac{\partial}{\partial t}\left(P/\rho^{\gamma}\right) + \underline{v}\cdot\underline{\nabla}\left(P/\rho^{\gamma}\right) = 0}$$

HW #2

show $\frac{dP}{dt} = \gamma P \, \underline{\nabla}\cdot\underline{v}$

eqn. of state

perfect $\begin{cases} \text{homogeneity} \\ \text{stationarity} \end{cases}$ $\left(P/\rho^{\gamma} = \text{const}\right)$

– "adiabatic equation of state"

4) Ohm's Law – *most* sensitive part of MHD (since controlled by *electrons*)

MHD variants differ primarily in Ohm's Law

- Hall MHD → Hall term
- EMHD → electron inertia
- Braginsky / drift MHD → ∇P terms

⋮ etc., etc.

- Ohm's Law ⟶ subtract moments on electron equations → electrons $\left(\underline{J} = nq(\underline{V_i} - \underline{V_e}) \right)$

Simple resistive MHD:

$$\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = \underset{\sim \nu_{ei}}{\eta}\, \underline{J}$$

$\sim \nu_{ei}$ → momentum transfer to ions ...

ideal MHD:

$$\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = 0$$

→ field "frozen into" fluid

⑤, ⑥, ⑦: Only 1 approximation:

$$\nabla\times\underline{B} = \frac{4\pi}{c}\underline{J} + \frac{1}{c}\cancel{\frac{\partial \underline{E}}{\partial t}}$$

$|\partial \underline{E}/\partial t| << |\underline{J}|$ → condition on ω?

$$\rightarrow \quad \omega \frac{\underline{v}B}{c} \overset{?}{<<} \frac{k}{c^{-1}} B$$

$$\Rightarrow |\underline{V}|(\omega/k)/c^2 << 1$$ is condition on ω.

→ Skeptic: "Does it Hang Together"?

i.e. is electric force negligible?

$$\rho \frac{d\underline{v}}{dt} = nq\,\underline{E} + \dots\dots$$

and $q \neq 0$, as

$$nq = \frac{\nabla\cdot E}{4\pi}$$

$$\underline{E} = -\frac{\underline{v}\times\underline{B}}{c}$$

so

$$nq\,\underline{E} = +\frac{(\underline{v}\times\underline{B})}{c}\,\underline{\nabla}\cdot\left(\frac{\underline{v}\times\underline{B}}{c}\right) \neq 0 \;!$$

but

$$\sim \frac{v^2}{c^2}\,B^2 k$$

$$\sim \frac{v^2}{c^2}\,(\underline{J}\times\underline{B}) \longrightarrow \text{negligible if } v^2/c^2 \ll 1.$$

Thus, yes indeed it does!

→ Putting it together:

$$\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = \eta\,\underline{J}, \qquad \underline{\nabla}\times\underline{E} = -\frac{1}{c}\frac{\partial\underline{B}}{\partial t}$$

⇒ the induction equation for $\underline{B}$ evolution ...

$$\boxed{\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times (\underline{V} \times \underline{B}) + \eta \nabla^2 \underline{B}}$$

- with momentum equation, defines MHD as problem of 2 coupled fluid fields (vector) — $\underline{V}(\underline{X}, t)$, $\underline{B}(\underline{X}, t)$ evolving simultaneously

↓

- useful and instructive to re-write induction equation

$$\nabla \times \underline{V} \times \underline{B} = -\underline{V} \cdot \underline{\nabla}\, \underline{B} + \underline{B} \cdot \underline{\nabla}\, \underline{V} - \underline{B}\, \underline{\nabla} \cdot \underline{V}$$

so
$$\frac{\partial \underline{B}}{\partial t} + \underline{V} \cdot \underline{\nabla}\, \underline{B} - \eta \nabla^2 \underline{B} = \underline{B} \cdot \underline{\nabla}\, \underline{V} - \underline{B}\, \underline{\nabla} \cdot \underline{V} .$$

This brings us to ......

→ What Does "MHD", as a system, really mean ....?

this is answered most clearly for the case of incompressible MHD ---

$\underline{\nabla}\cdot\underline{V} = 0$ → defines equation of state
→ sets P_{total} field

$$\left(\omega/k \ll \underset{\text{sound}}{C_s},\ V_{MS}\right)$$

↳ magnetosonic

$$\nabla\cdot\left\{\frac{\partial\underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V} = -\frac{\underline{\nabla}}{\rho}\left(P+\frac{B^2}{8\pi}\right) + \frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi\rho}\right\}$$

$$\frac{d\rho}{dt} = -\rho\,\underline{\nabla}\cdot\underline{V} = 0$$

so $\rho \rightarrow$ constant ρ_0 (can relax to slow variation)

$$\nabla^2\left[\left(P+\frac{B^2}{8\pi}\right)/\rho_0\right] = \underline{\nabla}\cdot\left(\frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi\rho_0} - \underline{V}\cdot\underline{\nabla}\,\underline{V}\right)$$

↑ total pressure

a la' Poisson's equation:

$$\frac{P+B^2}{8\pi} = -\int\frac{d^3x'}{4\pi|x-x'|}\left\{\underline{\nabla}\cdot\left(\frac{\underline{B}\cdot\underline{\nabla}'\underline{B}}{4\pi\rho_0} - \underline{V}\cdot\underline{\nabla}'\underline{V}\right)\right\}$$

solves for: P_{tot} field → eliminates eqn. state.

$p^* = p_{tot}$

so

$$\boxed{\begin{aligned} \frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V} &= -\underline{\nabla}\left(\frac{p^*}{\rho_0}\right) + \frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi\rho_0} \\ \frac{\partial \underline{B}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{B} - \eta\,\nabla^2\underline{B} &= \underline{B}\cdot\underline{\nabla}\,\underline{V} \end{aligned}}$$

with $\nabla\cdot\underline{V} = 0$, constitute equations of incompressible MHD.

→ Rather clearly, this system is one of two, dynamically coupled, evolving vector fields $\underline{V}(\underline{x},t)$, $\underline{B}(\underline{x},t)$.

→ Compressible MHD is really a problem in 3 fields, two of which are vectors

i.e.

$$\begin{cases} \underline{V}(\underline{x},t) & \rightarrow \text{fluid velocity} \\ \underline{B}(\underline{x},t) & \rightarrow \text{magnetic field} \\ S(\underline{x},t) & \rightarrow \text{entropy} \Rightarrow \text{energy density} \end{cases}$$

i.e. scalar equation of state provides 3rd field.

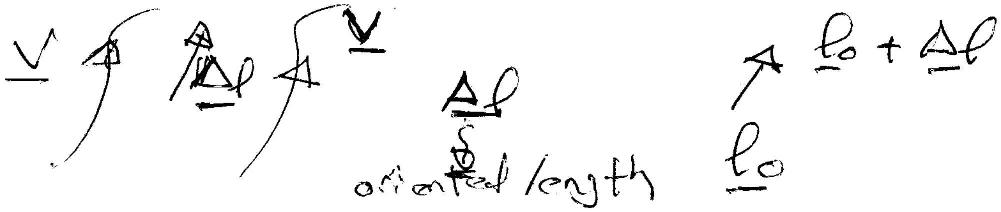
→ Key Question: How closely coupled are $\underline{V}$, $\underline{B}$??

⇒ **the** key physics element in MHD .....

⇒ $\boxed{\text{Frozen-in Law , Flux Freezing}}$

① Frozen-in Law

- consider a (for the moment, passive) vector field:
  - frozen into flow $\underline{V}(\underline{x}, t)$
  - consisting of oriented, flexible strands



{ i.e. massless rubber strands in flow

How does $\underline{\Delta l}$ evolve?

in dt, $d(\underline{\Delta l}) = (\underline{V}(\underline{l_0} + \underline{\Delta l}) - \underline{V}(\underline{l_0}))\, dt$

$$= \underline{\Delta l} \cdot \nabla \underline{V} \quad dt$$

$$\therefore \quad \frac{d(\underline{\Delta l})}{dt} = \underline{\Delta l} \cdot \underline{\nabla}\, \underline{V}$$

i.e. $\frac{d}{dt}\underline{\Delta \ell} = \underline{\Delta \ell} \cdot \underline{\underline{S}}$

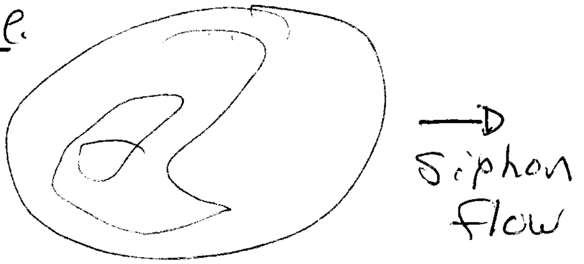
$$\left\{ \begin{aligned} & \frac{d}{dt}(\Delta \ell)_i = \underline{\Delta \ell}_j \cdot S_{ij} \\ & S_{ij} = \frac{1}{2}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) \end{aligned} \right.$$

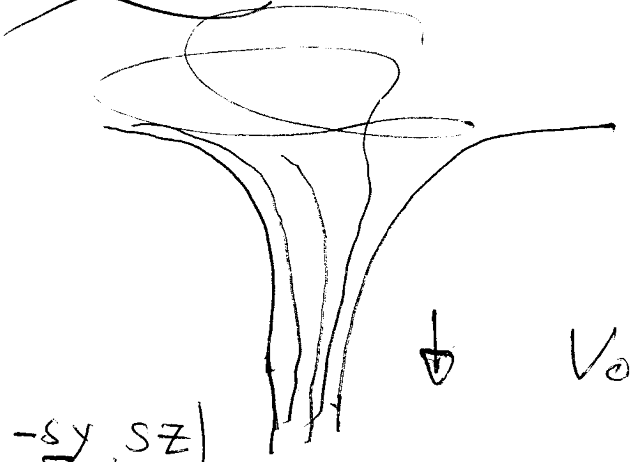
→ strain rate tensor

says that → $\underline{\Delta \ell}$ strands orient along strain

→ strain extends strands .....

i.e.





$\underline{V} = V_0\left(-\frac{sx}{2}, -\frac{sy}{2}, sz\right)$

plausible to say that $\underline{\Delta \ell}$ "frozen onto" the flow.

Now, if $\eta \to 0$, ... in MHD

$$\frac{\partial \underline{B}}{\partial t} + \underline{V} \cdot \underline{\nabla}\, \underline{B} = \underline{B} \cdot \underline{\nabla}\, \underline{V} - \underline{B}\; \underline{\nabla} \cdot \underline{V}$$

$$-\underline{\nabla} \cdot \underline{V} = +\frac{1}{\rho} \frac{d\rho}{dt}$$

$$\frac{\partial}{\partial t} \underline{B} + \underline{V} \cdot \underline{\nabla}\, \underline{B} = \frac{d\underline{B}}{dt} = \underline{B} \cdot \underline{\nabla}\, \underline{V} + \frac{\underline{B}}{\rho} \frac{d\rho}{dt}$$

$$\frac{1}{\rho} \frac{d\underline{B}}{dt} - \frac{\underline{B}}{\rho^2} \frac{d\rho}{dt} = \frac{\underline{B}}{\rho} \cdot \underline{\nabla} V$$

$$\therefore \quad \boxed{\frac{d}{dt}\left(\frac{\underline{B}}{\rho}\right) = \frac{\underline{B}}{\rho} \cdot \underline{\nabla}\, \underline{V}}$$

→ $\underline{\underline{B}/\rho}$ obeys same equation as $\underline{\Delta l}$!

→ $\underline{B}/\rho$ is frozen into flow field $\underline{V}(\underline{x}, t)$

Note: → $\underline{B}/\rho$ is **not** passive → due $\underline{J} \times \underline{B}$ force.

→ $\underline{B}$ determines flow, while frozen into it!

→ (essence of coupling problem)

- For $\underline{\nabla}\cdot\underline{V} = 0$, $\underline{B}$ frozen in
- if $\eta \neq 0$, freezing in is broken ----

i.e. $$\frac{d}{dt}\left(\frac{\underline{B}}{\rho}\right) - \underbrace{\frac{\eta}{\rho}\nabla^2\underline{B}}_{\text{form of frozen evolution broken}} = \frac{\underline{B}}{\rho}\cdot\underline{\nabla}\,\underline{V}$$

→ Observe: → this motivates attention to resistivity in MHD above other dissipations ν, χ, etc..

→ $\eta \Rightarrow \underline{B}$ diffusion $\sim \eta\nabla^2$]

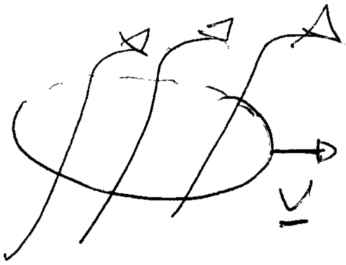
∴ decoupling of $\underline{V}$, $\underline{B}$ occurring on small scales

⇒ motivates 'magnetic reconnection' as study of singularity dynamics in MHD. [illegible] where is freezing in broken]

→ A Word to the Wise: In modelling, describing complex dynamics in MHD (i.e. MHD turbulence, dynamos, etc.) always think carefully about frozen-in law ...

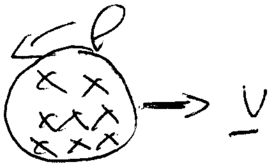
→ Closely Related: Flux Freezing

- consider flux thru surface in flow



→ i.e. imaginary loop drawn on flow field...

$$\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times \underline{V} \times \underline{B}$$

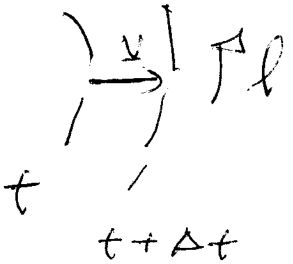


$$\underline{\Phi} = \int \underline{B} \cdot d\underline{s}$$

$$\frac{d\Phi}{dt} = \overset{①}{\underbrace{\int d\underline{s} \cdot \frac{\partial \underline{B}}{\partial t}}_{\text{change in } \underline{B}}} + \overset{②}{\underbrace{\int \frac{d\underline{s}}{dt} \cdot \underline{B}}_{\text{motion of loop...}}}$$

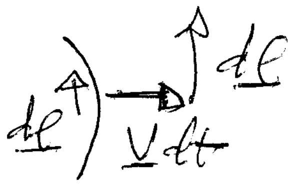
$$① = \int d\underline{s} \cdot \underline{\nabla} \times (\underline{V} \times \underline{B})$$

$$= \oint d\underline{\ell} \cdot (\underline{V} \times \underline{B})$$



$$ds = \underline{V}\,\Delta t \times d\underline{\ell}$$

For ②



$$\underline{\underline{\text{so}}} \quad d\underline{s} = \underline{V} dt \times d\underline{\ell}$$

↳ { change in $\underline{s}$ in dt.

$$dt \left(d\Phi_{②} / dt \right)$$

$$② dt = \int (\underline{V} dt \times d\underline{\ell}) \cdot \underline{B} = d\Phi$$

$$\left.\frac{d\Phi}{dt}\right)_{②} = \int (\underline{V} \times d\underline{\ell}) \cdot B = -\int d\underline{\ell} \cdot (\underline{V} \times \underline{B})$$

so $\frac{d\Phi}{dt} = ① + ②$

$= 0$

so ⇒ magnetic flux invariant ⟺ cancellation

⇒ in absence of resistivity, flux thru surface in flow is invariant, or frozen in

⇒ no surprise: $\underline{B}$ frozen in ⇒ Φ frozen in

⇒ analogue in hydro: Circulation (Kelvin's Thm.)

$$\Gamma_c = \oint \underline{v}\cdot d\underline{\ell} = \int d\underline{a}\cdot\underline{\omega} \qquad \omega = \underline{\nabla}\times\underline{v}$$

In inviscid hydro, $(\nu \to 0)$ circulation Γ_c is conserved.

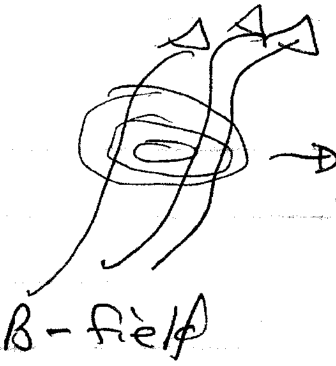
Exercise: Prove this!
Note relation between $\underline{\omega}$ equation and $\underline{B}$ eqn. Assume $\rho = \text{const}$, $\underline{g} = 0$

Extra Credit: ① Discuss the extension to the case where $\rho \neq \text{const}$.
② what is 'frozen in' for Vlasov plasma?

⇒ What Does "Freezing" Mean?

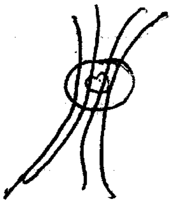
⇔ can relate field evolution in a flow to density evolution, since $\underline{B}/\rho$ is "frozen in".

Application:



Star formation

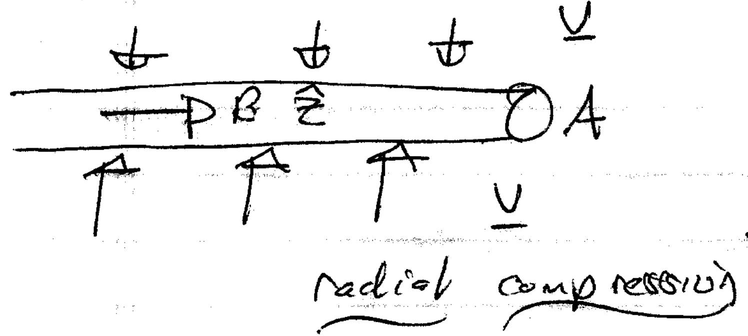
→



(?)

Simple Cases → How does $\underline{B}$ change in a flow? (what is α s/t $B \sim \rho^{\alpha}$?)

i)



Cylinder of cross-sectional area A squeezed in radial flow. $\underline{B} \equiv B\hat{z}$

2 ways:

$$\frac{d}{dt}(B\hat{z}/\rho) = \frac{B\hat{z}\cdot\underline{\nabla}\,\underline{V}}{\rho}$$

$\underline{V} = v\hat{r}$

$\Rightarrow \underline{V} \perp \underline{B}$

$$= 0$$

so $\underline{B}/\rho$ = const

Now: $\rho A L$ = const so $B \sim A^{-1}$

$\rho \sim A^{-1}$

(L const.)

or Flux Frozen: $BA = \Phi$ = const.

ρAL = const = M

L const.

$BA \sim \Phi_0$, $B \sim A^{-1}$

$\rho A \sim M_0$, $\rho \sim A^{-1}$

so $B \sim \rho^{(1)} \Rightarrow B/\rho \sim$ const!

$V = V(z)\hat{z}$ - compressible!

ii.) $\xrightarrow{} B\hat{z}$ i.e. stretch, 1D

here $\frac{\underline{B}}{\rho} \cdot \underline{\nabla}\, \underline{V} \neq 0$, but easier to work with $\underline{B}$ than $\underline{B}/\rho$

$$\frac{\partial \underline{B}}{\partial t} + \underline{V} \cdot \underline{\nabla}\, \underline{B} = \underline{B} \cdot \underline{\nabla}\, \underline{V} - \underline{B}\, \underline{\nabla} \cdot \underline{V}$$

$$= B \frac{\partial}{\partial z} V(z) - B \frac{\partial V(z)}{\partial z}$$

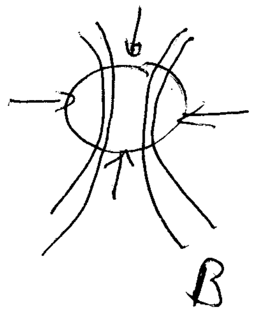
$$= 0 \quad !$$

For ρ, $\frac{d\rho}{dt} = -\rho \, \underline{\nabla} \cdot \underline{V} = -\rho \frac{\partial V_z}{\partial z}$

here $\underline{B}$ invariant, ρ changes

i.e. $B \sim \rho^{(0)}$

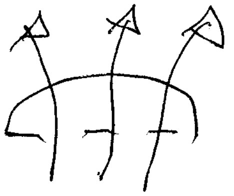
iii.)



collapsing sphere: $\underline{V} = v \hat{r}$

(i.e. $\Phi = 0$ total sphere)

consider hemispherical surface (i.e. mushroom cap)



$\Phi \sim B R^2 \sim$ const

$M \sim \rho R^3 \sim$ const.

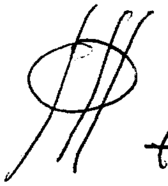
$\Rightarrow B \sim r^{-2}$
$\rho \sim r^{-3}$
$\Rightarrow B/\rho^{2/3} \sim$ const.

why the scaling? ⇔ why of interest?

→ "implosion"
gravitational collapse } problems sensitive to
equation of state of material collapsing

If: $p \to p_{tot} = p + \frac{B^2}{8\pi}$

$$p = p_0 (\rho/\rho_0)^{\gamma}$$



then natural to ask: Can one write $B^2 = B^2(\rho)$ and thus extend equation of state to encompass magnetic pressure contribution?

Proceed via flux-freezing!

$$B \sim \rho^{2/3} \Rightarrow B^2 \sim \rho^{4/3}$$

so p_{B^2} has "$\gamma_{eff} = 4/3$". This resembles equation of state for degenerate gas (see Handouts I).
(exclusion)

⇒ More on this in discussion of flux freezing and virial theorems ...

skip

→ Pragmatic Question: Is flux 'frozen' during star formation? ↔ Does resistivity matter?

$\eta \sim \frac{4 \times 10^6}{T_{eV}^{3/2}}$ cm²/sec. (Spitzer)

start → collapse → protostar

$n \sim 1$ atom/cm³ → $\rho \sim 1$ g/cm³ $n \sim 10^{24}$ atm/cm³ (related N_A)

but

$B / \rho^{2/3} \sim$ const

$\Rightarrow B / B_0 \sim (10^{24})^{2/3} \sim 10^{16}$! huge amplification

so $B_0 \sim 10^{-6}$ G, characteristic of ISM

$\Rightarrow B \sim 10^{10}$ G in protostar

$\therefore P_{B^2} \sim 10^{19}$ erg/cm³ $(P_{B^2} \sim B^2/8\pi)$

but P_{Th} for normal star $\sim 10^{14}$ erg/cm³

$P_{B^2} \gg P_{Th}$! ! $\Rightarrow$ clearly flux-freezing is bad assumption

→ In terms of time scales:

$$\underset{①}{\frac{\partial B}{\partial t}} = \underset{②}{\underline{\nabla} \times (\underline{v} \times \underline{B})} + \underset{③}{\eta \nabla^2 B}$$

$$\frac{1}{T_{collapse}} \sim \frac{1}{T_{dynamic}} + \underset{1/T_{diffn.}}{\underset{\simeq}{\frac{\eta}{L^2}}}$$

3 scales, 2 balance

i.e. ① & ② ③ negligible

① & ③ ② negligible.

if $T_{collapse} \ll T_{diffn} \rightarrow \{$ flux frozen, OK

$T_{collapse} \gg T_{diffn} \rightarrow \{$ must consider diffusion
freezing invalid

N.B.: In star formation, $T_{coll.} \ll T_{diffn}$

but ISM has large neutral component

Plasma-neutral drag sets dissipation

→ Ambipolar diffusion.

⇒ **Conservation Laws in MHD** — requisite for theory

- here discuss: conservation $\left\{\begin{array}{l}\text{momentum}\\ \text{energy}\\ \text{angular momentum}\end{array}\right.$

and virial theorems

> · **Momentum** → key: construct evolution of momentum density

have: · $$\rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V}\right) = -\underline{\nabla}\left(P+\frac{B^2}{8\pi}\right) + \frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi} + \underbrace{\rho\underline{g}}_{\text{body force}}$$

· $$\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{V}) = 0$$

⇒ → $$\underbrace{\frac{\partial}{\partial t}(\rho\underline{V})}_{\substack{\text{momentum}\\ \text{density}}} + \nabla\cdot\Big(\underbrace{\rho\,\underline{V}\,\underline{V}}_{\substack{\text{Reynolds stress}\\ \text{tensor}}}\Big) = -\nabla\left(P+\frac{B^2}{8\pi}\right) + \underline{\nabla}\cdot\underbrace{\frac{\underline{B}\,\underline{B}}{4\pi}}_{\substack{\text{Maxwell stress}\\ \text{tensor}}} + \rho\underline{g}$$

$$\underline{T}_R \equiv \rho\,\underline{V}\,\underline{V}$$

$$T_B = \frac{B^2}{8\pi}\underline{\underline{I}} - \frac{\underline{B}\,\underline{B}}{4\pi}.$$

thus re-write:

$$\frac{\partial}{\partial t}(\rho\underline{V}) = -\underline{\nabla}\cdot\underbrace{\underline{\underline{T}}}_{\text{stress tensor}} + \rho\underline{g}$$

where

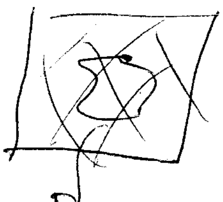
$$\underline{\underline{T}} = \left(p + \frac{B^2}{8\pi}\right)\underline{\underline{I}} + \frac{\underline{B}\,\underline{B}}{4\pi} - \rho\underline{V}\,\underline{V}$$

$$T_{ij} = \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} + \underset{\mapsto \text{ a.la Gaussian surface}}{\frac{B_i B_j}{4\pi}} - \rho V_i V_j$$

Then, if consider a 'blob' of $\left\{\begin{array}{l}\text{plasma}\\ \text{magnetofluid}\end{array}\right.$:

surface S — crucial

$$\underset{\substack{\downarrow\\ \text{momentum}}}{\frac{\partial}{\partial t}\underline{P}} = \int d^3x \, \frac{\partial}{\partial t}\overset{\substack{\text{momentum density}\\ \downarrow}}{(\rho\underline{V})}$$



blob enclosed by arbitrary, non-dynamical surface

$$= -\int d^3x \, \underline{\nabla}\cdot\underline{\underline{T}} + \underset{\mapsto \text{ net body force}}{\int d^3x \, \rho\underline{g}}$$

$$= -\int d\underline{S}\cdot\underline{\underline{T}} + \int d^3x \, \rho\underline{g}$$

So, apart from volume integrated body force,

$$\frac{\partial\underline{P}}{\partial t} = -\int d\underline{S}\cdot\underline{\underline{T}} \qquad \left\{\begin{array}{l}\text{change in momentum set}\\ \text{by stress on}\\ \text{surface of blob}\end{array}\right.$$

$$\underline{\underline{T}} = \left(p + \frac{B^2}{8\pi}\right)\underline{\underline{I}} - \frac{\underline{B}\,\underline{B}}{4\pi} + \rho\underline{V}\,\underline{V}$$

Thus, can identify ways momentum is lost by the blob:

→ $\underline{\underline{T}}_R \cdot d\underline{S} = -(\rho \underline{V})\underline{V} \cdot d\underline{S}$ → flux of momentum density thru surface

→ $-\underline{\underline{T}}_{P_{tot}} \cdot dS = -\left(P + \frac{B^2}{8\pi}\right) \cdot d\underline{S}$ → pressure (total) force on surface, in $-d\underline{S}$ direction

$-\underline{\underline{T}}_{\substack{Mag\\ten}}\, d\underline{S} = \frac{\underline{B}}{4\pi}\,\underline{B}\cdot d\underline{S}$ → magnetic tension force in $+\underline{B}$ direction, piercing surface

$$\sim \underbrace{(\underline{B}\cdot d\underline{S})}_{\substack{\#\text{ of lines thru } d\underline{S}\\ \text{outward}\\ \text{flux}}}\ \frac{\underline{B}}{4\pi}$$

→ tension of $\underline{B}/4\pi$, per line,

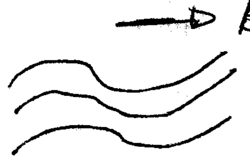
i.e.

→ Note that $\boxed{\text{magnetic tension}}$ is independent of sign of $\underline{B}$ (as it should, tension is, strictly speaking, a dyad $\updownarrow$, not $\uparrow$)

→ tension field $\sim \underline{B}\,\underline{B}$

→ can make obvious analogy between strings (see next) and field lines

(strings/flux)



$\#\text{strings}/\text{area} = B$

$\sigma = \rho/B$ → mass per length, of string

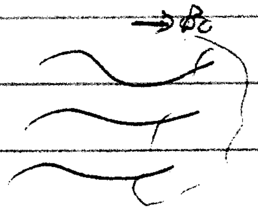
Alfven wave $T = B/4\pi$

$$\left.\begin{aligned} v_{ph}^2 &= T/\sigma \\ &= B^2/4\pi\rho_0 \\ &= v_A^2 \end{aligned}\right]$$

P3a.

strings ⇔ Field lines

$\# \text{ Strings} \sim \Phi$



$$\frac{\# \text{ Strings}}{\text{Area}} = B$$

need Mass/Length.)

$\therefore \quad \sigma \sim \rho/B \Rightarrow$ mass/length

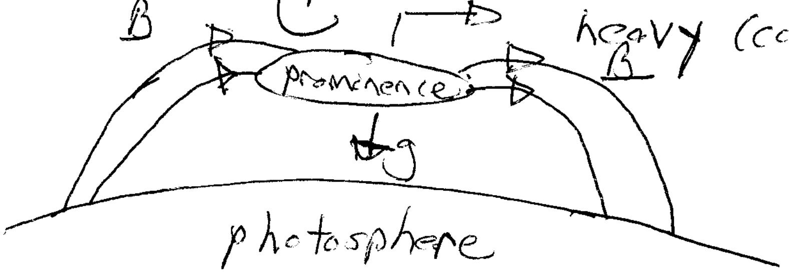
$T = B/4\pi$

$\sigma = \rho/B$

$T/\sigma = V^2 = B^2/4\pi\rho \Rightarrow$ Alfven speed

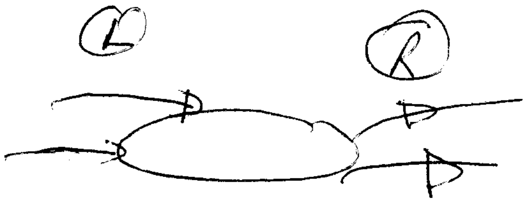
→ example: {Solar prominence (see cover of Kulsrud)

heavy (cool) → requires support against gravity



{prominences often associated with radiative condensation

i.e.



L ⇒ #lines/area = $\underline{B}\cdot d\underline{S} < 0$ (inward)

force/Line is toward

∴ $\underline{F}_{Ⓛ}$ → toward upper left

R ⇒ #lines/area = $\underline{B}\cdot d\underline{S} > 0$

f/Line is toward upper right

$F_{Ⓡ}$ → toward upper right

∴us ⟹ prominence supported by magnetic tension (cf hammock — string)

→ squashing $\underline{B}$ → support by magnetic pressure, too ....

→ The Skeptic : "what of EM Momentum ?"

$$\underline{P}_{EM} = \underline{E}\times\underline{B}/4\pi c$$

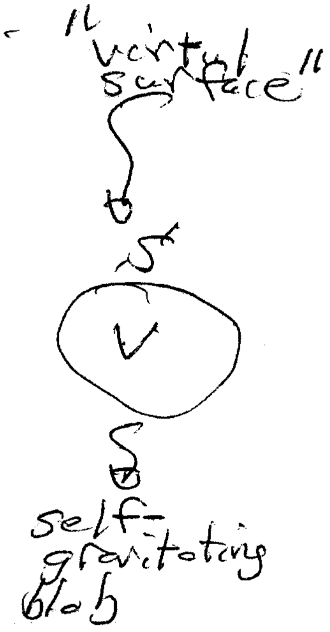
$$E \sim \frac{VB}{c} \Rightarrow P_{EM} \sim \underline{(\rho V)}\, B^2/4\pi\rho c^2$$

$$\sim \rho V \left(V_A^2/c^2\right) \ll 1$$

n.b. obviously important in relativistic and EMHD, for $V_A \ll c$..

→ Angular Momentum → read Kulsrud ---- "virtual surface"

→ **Energy**



$$\text{Now energy} : E = \overset{\text{kinetic}}{E_V} + \overset{\text{thermal}}{E_p} + \overset{\text{magnetic}}{E_B} + \overset{\text{gravity}}{E_g}$$

$$E = \int_V d^3x \left[\underset{\text{kin}}{\frac{1}{2}\rho V^2} + \underset{\text{Th.}}{\frac{p}{\gamma-1}} + \underset{\text{Mag}}{\frac{B^2}{8\pi}} + \underset{\text{grav}}{\frac{\rho\phi}{2}} \right]$$

$$\text{where} \begin{cases} \underline{g} = -\underline{\nabla}\phi \\ \nabla^2\phi = 4\pi G\rho \end{cases}$$

i.e. $\underline{g}$ evolves self-consistently (not "constant")

N.B. Problem : Jeans Instability

→ Calculate the growth rate of density perturbations in an un-magnetized, self-gravitating fluid

→ repeat in 1D, using Vlasov equation

→ Where does E_p come from?

Consider work to compress plasma/fluid, i.e.

$$dW = -p\,dV$$

$$\Delta E = -\int_0^{\rho_0} P(\rho)\, d(1/\rho) = \int_0^{\rho_0} (\rho/\rho_0)^{\gamma} P_0 \frac{d\rho}{\rho^2}$$

$$= \frac{P_0}{\rho_0(\gamma-1)} \qquad \Rightarrow \underset{\substack{\downarrow \\ \text{energy density}}}{\mathcal{E}} = \rho_0 \Delta E = \frac{P_0}{(\gamma-1)}$$

→ for energy balance, crank it out, using MHD equations ....

$$\frac{dE}{dt} = \overset{①}{\frac{dE_V}{dt}} + \overset{②}{\frac{d}{dt}E_p} + \overset{③}{\frac{d}{dt}E_B} + \overset{④}{\frac{d}{dt}E_g}$$

go to 34 explain

① $$\frac{d}{dt}E_V = \int d^3x \frac{\partial}{\partial t}\left(\frac{\rho V^2}{2}\right)$$

$$= \int d^3x \left[V^2 \frac{\partial \rho}{\partial t} + \rho \frac{\partial \underline{v}}{\partial t} \cdot \underline{V} \right] d^3x$$

→ ibp leaves $\underline{S.T.}$ and cancels 2nd.

$$= \int d^3x \left[-\frac{\underline{V}^2}{2} \nabla\cdot(\rho \underline{V}) - \underline{V}\cdot\rho(\underline{V}\cdot\underline{\nabla}\underline{V}) - \underline{V}\cdot\underline{\nabla}P + \underline{V}\cdot(\underline{J}\times\underline{B}) - \rho\underline{V}\cdot\underline{\nabla}\phi \right]$$

i.e. $$\int -\frac{V^2}{2}\nabla\cdot(\rho\underline{V}) = -\frac{V^2}{2}\rho\underline{V}\Big| + \int \underbrace{(\underline{V}\cdot\underline{\nabla}\,\underline{V})}_{\text{cancels 2nd term in } \frac{dE_V}{dt}}\cdot\rho\underline{V}$$

---

② $$\frac{d}{dt}E_p = \int\frac{d^3x}{\gamma-1}\frac{\partial p}{\partial t}$$

Now eqn. state $\Rightarrow$ $$\frac{1}{p}\frac{dp}{dt} + \frac{\gamma}{\rho}\frac{d\rho}{dt} = 0$$

$$\left[\frac{1}{(p/\rho^\gamma)}\frac{d}{dt}\left(p/\rho^\gamma\right) = 0\right]$$

and $$\frac{1}{\rho}\frac{d\rho}{dt} = -\underline{\nabla}\cdot\underline{V}$$

$$\Rightarrow \frac{\partial p}{\partial t} = -\underline{V}\cdot\underline{\nabla}p - \gamma p\,\underline{\nabla}\cdot\underline{V}$$

So $$\frac{d}{dt}E_p = \frac{-1}{(\gamma-1)}\int d^3x\left(\underline{V}\cdot\underline{\nabla}p + \gamma p\,\underline{\nabla}\cdot\underline{V}\right)$$

$$= -\int d^3x\left[\underbrace{\frac{\gamma}{\gamma-1}\nabla\cdot(p\underline{V})}_{\substack{\text{yields a}\\ \text{surface}\\ \text{term}}} - \underbrace{\underline{V}\cdot\underline{\nabla}p}_{\substack{\text{cancels}\\ \underline{V}\cdot\underline{\nabla}p \text{ term}\\ \text{in } \frac{dE_V}{dt}}}\right]$$

expect similar relation between $\underline{J}\times\underline{B}$ and $\frac{\partial}{\partial t}B^2$ .... so...

③ $$\frac{d}{dt} E_B = \frac{1}{4\pi}\int d^3x \; \underline{B}\cdot\frac{\partial \underline{B}}{\partial t}$$

$$= \frac{1}{4\pi}\int d^3x \; \underline{B}\cdot\left(\underline{\nabla}\times\underline{v}\times\underline{B}\right)$$ by induction eqn.

ⓐ ⓑ

$$= -\int d^3x \left\{ \underline{\nabla}\cdot\left[\frac{\underline{B}\times(\underline{v}\times\underline{B})}{4\pi}\right] - \frac{\left(\underline{\nabla}\times\underline{B}\right)}{4\pi}\cdot\left(\underline{v}\times\underline{B}\right)\right\}$$

$\int$ surface term ($\rightarrow$ Poynting)

$\int \downarrow$ $\underline{J}\cdot\underline{v}\times\underline{B}$

ⓑ $$= \int d^3x \; \underline{J}\cdot\left(\underline{v}\times\underline{B}\right) = -\int d^3x \; \left(\underline{J}\times\underline{B}\right)\cdot\underline{v}$$

$\downarrow$ cancels $\underline{v}\cdot\underline{J}\times\underline{B}$ term in dE_v/dt

which leaves:

④ $$\frac{dE_g}{dt} = \frac{1}{2}\int d^3x \left(\phi\frac{\partial\rho}{\partial t} + \rho\frac{\partial\phi}{\partial t}\right)$$

$$= \frac{1}{2}\int d^3x\; \phi\frac{\partial\rho}{\partial t} + \int d^3x \frac{\nabla^2\phi}{8\pi G}\frac{\partial\phi}{\partial t}$$ ibp $\Rightarrow$

$$= \frac{1}{2}\int d^3x\,\phi\frac{\partial\rho}{\partial t}\,d^3x + \int\frac{\phi}{8\pi G}\nabla^2\frac{\partial\phi}{\partial t}\,d^3x$$

so

$$\frac{dE_g}{dt} = \frac{1}{2}\int \phi \frac{\partial \rho}{\partial t} d^3x + \frac{1}{2}\int d^3x\, \phi \frac{\partial \rho}{\partial t}$$

$$= \int d^3x\, \phi \frac{\partial \rho}{\partial t} = -\int d^3x\, \phi\, \nabla\cdot(\rho \underline{V})$$

$$= + \int d^3x\, \rho \underline{V}\cdot \underline{\nabla}\phi$$

cancels $-\rho\, \underline{V}\cdot\nabla\phi$ in $\frac{dE_V}{dt}$

$$+ -\int d\underline{S}\cdot \rho\phi\, \underline{V}$$

Note: $-\underline{V}\cdot\underline{\nabla} P$; $\underline{V}\cdot(\underline{J}\times\underline{B})$; $-\rho\,\underline{V}\cdot\underline{\nabla}\phi$; $\underline{V}\cdot\rho\,\underline{V}\cdot\underline{\nabla}\underline{V}$

terms <u>all</u> <u>cancel</u> in dE_V/dt !

Now, adding up all <u>4</u> pieces ⇒

$$\boxed{\frac{d}{dt}E = -\int d\underline{S}\cdot\left[\rho\underline{V}\frac{V^2}{2} + \frac{\gamma}{\gamma-1}P\underline{V} - \frac{(\underline{V}\times\underline{B})\times\underline{B}}{4\pi} + \rho\underline{V}\phi\right]}$$

<u>i.e.</u> not surprisingly, only survivors are surface terms . . . . ⇒ in ideal MHD, only change in energy of blob involves boundary . . .

i.e. have:

$$\frac{dE}{dt} = -\int d\underline{S}\cdot\left[\overset{①}{\rho\underline{V}\frac{V^2}{2}} + \overset{②}{\frac{\gamma}{\gamma-1}p\underline{V}} - \overset{③}{\frac{(\underline{V}\times\underline{B})\times\underline{B}}{4\pi}} + \overset{④}{\rho\underline{V}\phi}\right]$$

① → kinetic energy loss via simple kinetic energy flow thru surface.

② → $-\frac{\gamma\underline{V}\cdot d\underline{S}\,p}{\gamma-1}$ ↔ outward flow of enthalpy

i.e. $-\frac{\gamma p\underline{V}\cdot d\underline{S}}{\gamma-1} = -\frac{p}{\gamma-1}\underline{V}\cdot d\underline{S} - p\underline{V}\cdot d\underline{S}$

why the γ? →

outward flow of thermal energy $\left(d\underline{S}\cdot\underline{V}\frac{p}{\gamma-1}\right)$ thru S

pdV work of blob on exterior

③ as $\underline{E} = -\frac{\underline{V}\times\underline{B}}{c}$ . . . →

so ③ $= d\underline{S}\cdot\frac{\underline{E}\times\underline{B}}{4\pi c}$ → loss of energy by Poynting flux

④ loss of gravitational potential energy due outflow from blob...

It's all clear!!

$$\left.\begin{array}{ll}\text{Eqns} & \text{Linear Waves}\\ \text{Physics} & \underline{\text{Virial Thm}}\\ \text{Freezing-in law} & \\ \text{Momentum} & \\ \text{Energy} & \end{array}\right\} = \text{ideal} \rightarrow \begin{array}{c}\text{resistive}\\ \&\\ \text{reconnection}\end{array}$$

This brings us to ....

→ <u>Virial Theorems in MHD</u>

- what is a virial theorem
- why yet another theorem?

→ Virial Theorems are:

- space/time averaged energy theorems
- "lumped parameter" relations for energies in complex, multi-element interacting systems
- useful for 'back-of-envelope' estimates, etc.
- logically extend the moment program:

$$\underbrace{f(\underline{x},\underline{v},t)}_{\text{phase space fluid}} \xrightarrow[\underline{v}\text{ moments}]{} \underbrace{n(\underline{x},t),\underline{v},T}_{\text{position space }\underline{\text{fluid}}} \xrightarrow[\underbrace{\text{virial integrals}}_{\text{space integrals}}]{} \underbrace{E_v,\ E_B,\ \text{etc.}}_{\#}$$

Before proceeding:

[¿ Can an isolated blob of MHD plasma confine itself without self gravity ?]

Easily answered by Virial Theorem....

Recall, for system of particles, Virial theorem derived by considering:

$$\frac{d}{dt}\left(\sum_i \underline{p}_i \cdot \underline{x}_i\right) = \sum_i \underline{p}_i \cdot \dot{\underline{x}}_i + \sum_i \dot{\underline{p}}_i \cdot \underline{x}_i$$

$\rightsquigarrow$ action

$$= \underset{\substack{\uparrow \\ \text{kinetic} \\ \text{energy}}}{2T} + \underset{\substack{\uparrow \\ \text{via Newton's Law}}}{\sum_i \left(-\frac{\partial U}{\partial \underline{x}_i}\right) \cdot \underline{x}_i}$$

Now, if $\sum_i \underline{p}_i \cdot \underline{x}_i$ bounded,

$$\left\langle \frac{d}{dt}\sum \underline{p}_i \cdot \underline{x}_i \right\rangle = \frac{1}{T}\int_0^T dt\, \frac{d}{dt}\left(\sum_i \underline{p}_i \cdot \underline{x}_i\right)$$

$$\underset{T \to \infty}{\longrightarrow} 0$$

so ...

$\rightarrow$ (First) Virial of system

$$2\langle T\rangle = \left\langle \sum_i \frac{\partial U}{\partial \underline{x}_i} \cdot \underline{x}_i \right\rangle$$

Further, if $U = U(\underline{x}_1, \underline{x}_2, \cdots \underline{x}_N)$

where $U(\alpha \underline{x}_1, \alpha \underline{x}_2, \cdots \alpha \underline{x}_N) = \alpha^{\overset{\text{power law}}{\textcircled{k}}} U(\underline{x}_1, \underline{x}_2, \cdots \underline{x}_N)$

(scaling $\Leftrightarrow$ structure of power-law potentials $\rightarrow$ i.e. H.O. $\rightarrow$ $k=2$, Coulomb $\rightarrow$ $k=-1$)

$\downarrow$ homogeneous function

$$\Rightarrow \boxed{2\langle T\rangle = k\langle U\rangle}$$

but of course!

$$T + U = \langle T\rangle + \langle U\rangle = E$$

then $\left(\frac{k}{2} + 1\right)\langle U\rangle = E$

$$\boxed{\langle U\rangle = \frac{2}{k+2} E}, \quad \boxed{\langle T\rangle = \frac{kE}{k+2}} \checkmark$$

check: $k=2$, $\langle U\rangle = \frac{1}{2}E$, $\langle T\rangle = \frac{1}{2}E$

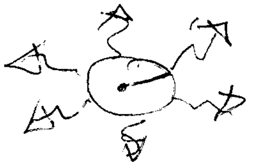
$k=-1$, $\langle T\rangle = -E$ ✓ $\Rightarrow$ (bounded motion only if total energy negative (i.e. bound state))

$(\Rightarrow E<0)$

Aside: Simplest realization of negative specific heat 'paradox', i.e.

 → consider 'blob' of self gravitating matter

$$E \sim -1/R$$

if radiation  → E decreases ⟷ R decreases

$\therefore \underbrace{(-E)}_{\text{kinetic energy}}$ increases $\Rightarrow \langle T \rangle$ increases

but $\langle T \rangle \sim$ temperature, so have cycle of: radiative cooling ⇒ temperature increase ?!

$$\Rightarrow \underbrace{c}_{\text{specific heat}} < 0 \;!?$$

In the days before the discovery of nuclear fusion this was thought to be what 'heated' stars. Kelvin, in particular, was a proponent.

Now, proceeding to full Virial theorem ...

→ Consider equation of motion

$$\underset{\text{momentum}}{\frac{\partial}{\partial t}(\rho v_i)} = - \frac{\partial}{\partial x_j} \underset{\underline{\text{full}}\text{ stress tensor}}{\overline{T}_{ij}}$$

$$T_{ij} = \rho v_i v_j + \left(p + \frac{B^2}{8\pi} \right) \delta_{ij} - \frac{B_i B_j}{4\pi} + \rho \phi \delta_{ij}$$

Now, recalling relation of virial to $\frac{d}{dt}(\underline{\rho} \cdot \underline{x})$
⇒ consider:

$$I_{ij} = \int d^3x \, \rho \, x_i x_j \qquad (\sim \text{moment of inertia})$$

↳ Virial theorem is for <u>tensor</u> --

and
$$\frac{d}{dt} I_{ij} = \int d^3x \, \frac{\partial \rho}{\partial t} x_i x_j$$

$$= - \int d^3x \, \frac{\partial}{\partial x_t} (\rho v_t) \, x_i x_j \qquad \underline{\text{continuity}}$$

integrating by parts assuming ρ compact (i.e. 'blob' of interest)

$$= \int d^3x \left[\rho x_i v_j + \rho x_j v_i \right]$$

so

$$\frac{d^2 I_{ij}}{dt^2} = \int d^3x \left[x_i \left(\frac{\partial}{\partial t} \rho v_j \right) + x_j \frac{\partial}{\partial t} (\rho v_i) \right]$$

but $$\frac{\partial}{\partial t}(\rho V_i) = -\frac{\partial}{\partial x_k} T_{ik}$$

⇒

$$\frac{d^2 I_{ij}}{dt^2} = -\int d^3x \left[x_i \frac{\partial T_{j,t}}{\partial x_t} + x_j \frac{\partial T_{i,t}}{\partial x_t} \right]$$

and integrating by parts, assuming { compact blob, no external linkage

⇒

$$\frac{d^2 I_{i,j}}{dt^2} = +\int d^3x \left[\delta_{i,t} T_{j,t} + \delta_{j,t} T_{i,t} \right]$$

$\partial x_i / \partial x_t = 0$ unless $i = t$

$$= +\int d^3x \left[T_{j,i} + T_{i,j} \right]$$

and as T_{ij} manifestly symmetric ⇒

$$\boxed{\begin{aligned} \frac{1}{2}\frac{d^2 I_{ij}}{dt^2} &= +\int d^3x\, T_{ij} \\ T_{ij} &= \rho V_i V_j + \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi} + \rho\phi\delta_{ij} \end{aligned}}$$

— tensor virial theorem.

note unlike simple pt particle example, time dependence remains.

Now, to make contact with notions of energy, etc., useful to contract the tensor

$$I = I_{i,i} = \text{tr}\, I_{i,j}$$

repeated indexes summed

tr (V.T.) ⇒

$$\text{tr}\,\frac{1}{2}\frac{d^2 I_{i,j}}{dt^2} = \frac{d^2}{dt^2}\left(\int d^3x\,\frac{\rho x^2}{2}\right)$$

$$= \text{tr}\int d^3x\left[\rho v_i v_j + \left(p+\frac{B^2}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi} + \rho\phi\,\delta_{ij}\right]$$

$$= \int d^3x\left[\rho v^2 + 3\left(p+\frac{B^2}{8\pi}\right) - \frac{B^2}{4\pi} + 3\rho\phi\right]$$

$$\therefore \quad I \equiv \int d^3x\,\rho x^2/2 \quad \Rightarrow$$

$$\boxed{\frac{d^2 I}{dt^2} = \int d^3x\left[\rho v^2 + 3p + \frac{B^2}{8\pi} + 3\rho\phi\right]}$$

→ Scalar Virial Theorem.

Now, first neglect self-gravitation ⇒

$$\frac{d^2 I}{dt^2} = \frac{d^2}{dt^2}\left(\int d^3x \, \frac{\rho x^2}{2}\right)$$

$$= \int d^3x \left[\rho V^2 + 3p + B^2/8\pi\right]$$

Now → can an isolated blob of MHD fluid confine itself ??

If 'self-confined' ⇒ $\frac{dI}{dt} \leq 0$

$\frac{d^2 I}{dt^2} \leq 0$

i.e. quiescent ⇒ $\dot{I}, \ddot{I} = 0$

stable pulsation ⇒ $\ddot{I} = -\Omega^2 I < 0$

but have $\ddot{I} = \int d^3x \left[\rho V^2 + 3p + B^2/8\pi\right]$

so even if $V^2 = 0$ (no fluid motion in blob) ⇒

$p > 0$, $B^2/8\pi > 0$ ⇒ $\ddot{I} > 0$!

∴ No → isolated blob can't confine itself.

More generally, noting that

$$E_V = \int d^3x\, \rho V^2/2$$

$$E_P = \int d^3x\, \frac{P}{\gamma-1} = \frac{3}{2}\int d^3x\, P \qquad (\text{gas})$$

$$E_B = \int d^3x\, \frac{B^2}{8\pi}$$

can write scalar Virial theorem in form:

$$\boxed{\frac{d^2 I}{dt^2} = 2E_V + 2E_P + E_B}$$

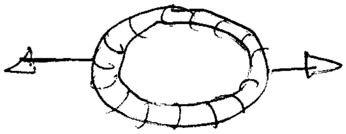
simple relation in terms energies.

Aside: → So, isolated blob can't confine itself

⇒ how is
- tokamak → B_T for
  - stability; not
  - transport
- or - better macro-confinement
- RFP → weak external B_T guide (negligible)

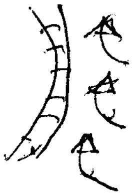
confined ?? Confinement by wall is unacceptable ...

Answer: → toroidal plasma tends to expand toroidally



→ held in place by $\left\{ \begin{array}{l} \text{conducting shell (often undesireable)} \ \text{or} \\ \text{"vertical field"} \end{array} \right.$

e.g.



→ additional external P_{Mag} to oppose toroidal expansion — vertical field

→ image currents in close-in conducting shell can do likewise

JET anecdote re: vertical field failure ...

Now, retaining self-gravitation:

$$T_{ij}\Big|_{gravity} = \rho\phi\,\delta_{ij} = 2\underbrace{\left(\frac{\rho\phi}{2}\right)}_{\mathcal{E}_{gravity}}\delta_{ij} .$$

to calculate:
$$\nabla^2\phi = 4\pi G\rho$$

$$\Rightarrow \quad \phi = -G\int d^3x \, \frac{\rho(x')}{|x-x'|}$$

so

$$T_{ij}\Big|_{\text{gravity}} = T\Big|_{\text{gravity}} \delta_{ij}$$

$\Rightarrow$

$$T = -\frac{G}{2}\int d^3x \int d^3x' \frac{\rho(\underline{x})\,\rho(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

$$= + E_{\text{gravitation}} \equiv E_g < 0$$

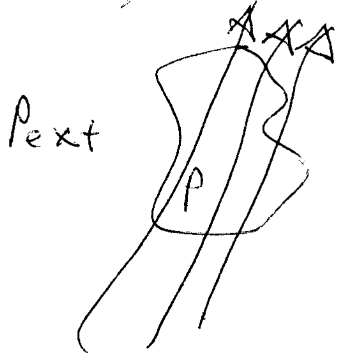
so scalar Virial theorem becomes, with gravity $\Rightarrow$

$$\boxed{\frac{1}{2}\frac{d^2 I}{dt^2} = 2E_V + 2E_P - |E_g| + E_B}$$

so with gravity, can have self-confining blob (no surprise...)

This brings us to another application of Virial theorems, namely proto-stellar cloud collapse....

- now, consider a plasma cloud/blob



- mass M, radius R
- threaded by B
- pressure P, external pressure P_0
- no bulk motion
- frozen flux

now, easy to show for $\ddot{\underline{I}} = 0$, $\underline{V} = 0$, must have:

$$2E_p - |E_g| + E_B = \underbrace{\int dA\, P_{ext}\,\hat{x}\cdot\hat{n} - \int dA\, \underline{x}\cdot\underline{\underline{T}}_B\cdot\hat{n}}_{\text{surface terms}}$$

$\int dA\, P_{ext}\,\hat{x}\cdot\hat{n}$ → external pressure.

$\int dA\, \underline{x}\cdot\underline{\underline{T}}_B\cdot\hat{n}$ → magnetic stress thru surface. (threading fields)

Now, can estimate:

$$M = \int \rho\, dV \quad \rightarrow \text{total mass}$$

$$E_p \cong C_s^2 M$$

$$|E_g| \cong \underbrace{\alpha}_{\text{form factor}} \frac{GM^2}{R}$$

For frozen flux, $\Phi \sim \pi R^2 B$

04 (both)

$$\tilde{80} \qquad E_B + \cancel{\int dA\, \underline{x}\cdot\underline{\underline{T}}_B\cdot\hat{n}} \sim \beta\,\Phi^2/R$$

$\Rightarrow$ have: (eliminating extraneous factors):

$$\boxed{R^2 P_{ext} \sim \left(\frac{\beta\Phi^2}{R} - \alpha\frac{GM^2}{R} + \frac{3}{2}c_s^2 M\right)}$$

P_{canic} $\Leftrightarrow$ stability

$\rightarrow$ scalar virial theorem for cloud....

Now: $$P_{ext} \sim \left(\frac{\beta\Phi^2}{R^3} - \alpha\frac{GM^2}{R^3} + \frac{3}{2}\frac{c_s^2 M}{R^2}\right)$$

$\rightarrow$ if $\Phi, G \rightarrow 0$ $\rightarrow$ need $P_{int} = P_{ext}$ for confinement.....

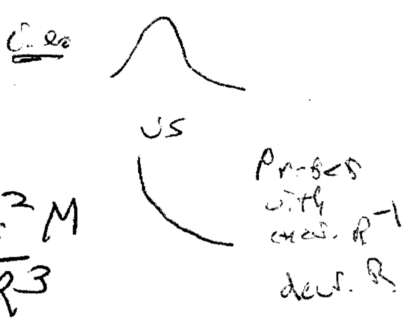
R_{max} → radius ~~max pressure~~ max pressure?

yes – unstable

no – stable

$\rightarrow$ if $\Phi = 0$

$$P_{ext} = -\alpha\frac{GM^2}{R^3} + \frac{3}{2}\frac{c_s^2 M}{R^2}$$

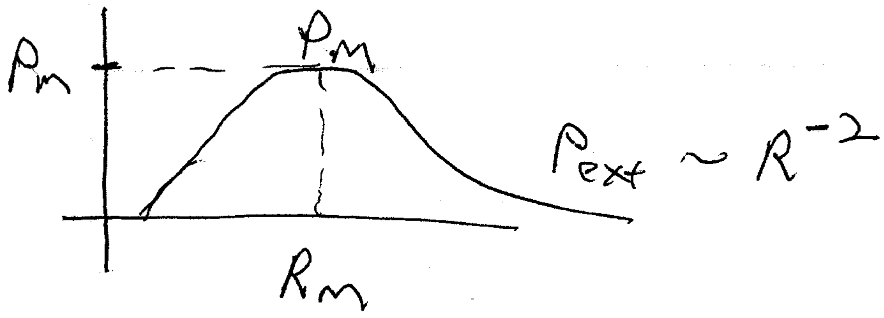


$$dP/dR = 0 \Rightarrow 3\alpha\frac{GM^2}{R^4} = 3\frac{c_s^2 M}{R^3}$$

$$R_{max} = GM\alpha/c_s^2$$

[Note: $\Rightarrow R_m^2 = (G\rho/c_s^2)^{-1/2} \Rightarrow L^2_{Jeans}$]

so P_{ext} needed



Φ $R < R_m$ → collapse

$R > R_m$ → sub crit ~bar

- $P > P_{max}$ → no equilibrium
- $R < R_{max}$ → P_{ext} must decrease to maintain equilibrium ⇒ instability to gravitational collapse!

i.e. smaller radius → less P_{ext} to confine → smaller radius ...

→ $\Phi \neq 0$ (magnetic field on ...) → note immediately that magnetic support scales similarly to gravitational attraction

⇒

$$P_{ext} \sim \left[\left(\beta \Phi^2 - \alpha G M^2 \right) / R^3 + \frac{3}{2} \frac{C_s^2 M}{R^2} \right]$$

so key point is: $\left(\beta \Phi^2 - \alpha G M^2 \right) \lesseqgtr 0$?!

$$\Rightarrow M \gtrless M_\Phi = \sqrt{\beta/\alpha}\, \Phi / G^{1/2}$$

→ $M < M_\Phi$ → magnetically subcritical mass for gravitational collapse

$M > M_\Phi$ → magnetically super-critical mass for collapse!

i.e. $M < M_\Phi$ → repulsive effects $\left\{\begin{matrix}\text{field}\\ \text{thermal}\end{matrix}\right\}$ pressure always win

$(M_\Phi^2 - M^2 > 0)$ → no ammount of external compression can induce indefinite contraction, IF flux remains frozen in

$M > M_\Phi$ → sufficient external pressure/compression can induce gravitational collapse, even if flux frozen in.

$(M_\Phi^2 - M^2 < 0)$

[Note: If kinetic energy contribution, NL Alfven waves can support cloud.]

For perspective, recall:

- (famous) Chandrasekhar Mass
  - $M > M_{Chandra}$ → collapse
  - $M < M_{Chandra}$ → no collapse

$M_{Chandra}$ derived for degenerate Fermi gas equation of state → $\gamma = 4/3$, instead of $\gamma = 5/3$.

– of flux – freezing $\Rightarrow$ $\Phi \sim B R^2$

$\rho R^3 \sim M$

$\Rightarrow B \sim R^{-2}$ $\Rightarrow$ $B \sim \rho^{2/3}$

$\therefore$ $B^2 \sim P_{Mag} \sim \rho^{4/3}$

$\Rightarrow$ <u>if</u> flux frozen, field obeys equation of state like Fermi gas

(i.e. flux freezing is akin to exclusion, albeit on field-lines-per-fluid-element)

$\therefore$ an analogue to Chandrasekhar mass seems quite plausible ....

Aside: Chandrasekhar Limit — Simple Derivation
(c.f.: Shapiro, Teukolsky)

→ suppose: N Fermions in star of radius R

∴ $n_{\text{Fermion}} \sim N/R^3$ density

∴ Vol./Fermion $\sim 1/n$ (Pauli exclusion)

$$\underbrace{p}_{\text{Fermion Momentum}} \sim \hbar/\Delta x \sim \hbar n^{1/3}$$

(Heisenberg Uncertainty)

⇒ Fermion energy (per Fermion) : $E_F = pc \sim \hbar c \frac{N^{1/3}}{R}$

replaces: (i.e. Thermal energy)

$\mathcal{E} \sim N E_F$

→ Baryon mass

Gravitational Energy (per Fermion) : $E_{Grav} \sim -\frac{GMm_b}{R}$

$M \sim N\ m_B$

Pressure → electron
Mass → Baryon

$$\therefore\ E = E_F + E_G$$
$$= \frac{\hbar c N^{1/3}}{R} - \frac{GNm_B^2}{R}$$

note: $E = E_F + E_G$

$$= \frac{\hbar c N^{1/3}}{R} - \frac{G N m_B^2}{R}$$

$E > 0 \Rightarrow$ decrease E, E_F by increasing R.

but as $E_F \downarrow$, electrons non-relativistic,
∴ $E_F \sim 1/R^2 \to$ eqbrm.

$E < 0 \Rightarrow$ decrease E without bound by decreasing $R \Rightarrow$ collapse.

∴ eqbrm: $\hbar c N^{1/3} = G N m_B^2$

$$N_{Max} = \left(\hbar c / G m_B^2\right)^{3/2} \sim 2 \times 10^{57} \quad \text{(proton)}$$

∴ $M_{Chandra} = N_{max} m_B \sim 1.5 M_\odot$

Supplement

→ Deriving MHD

→ MHD is derived from 2-fluid equations

- first discuss 2 fluid derivation from Boltzmann
- then discuss reduction to one-fluid MHD (i.e. approximations/limitations — especially in Ohm's Law)

→ deriving fluid equations

Have in general, Boltzmann eqn

$$\overset{①}{\frac{\partial f}{\partial t}} + \overset{②}{\underline{v}\cdot\underline{\nabla} f} + \overset{③}{\frac{q}{m}\left(\underline{E} + \frac{\underline{v}\times\underline{B}}{c}\right)\cdot\underline{\nabla} f} = \overset{④}{\underbrace{C(f)}_{\text{collision operator}}}$$

and can assign time scale

① ↔ ω → frequency

② ↔ $v_{Th}/L_{\parallel}$

↳ <u>relevant</u> parallel scale

③ $\frac{q}{m}\frac{E}{\Delta V}$ $\Delta V \sim V_{Th}$ → non-resonant
$\Delta V \sim \Delta V_{Tr}$ → resonant

NL scattering rate (→ small, usually)

④ ν_{eff} - collision frequency.

For "fluid description", need:

→ $\nu_{eff} > V_{Th}/L_{\parallel}$

i.e short mean free path limit

or

→ $\omega > V_{Th}/L_{\parallel}$ → a/a' gyrokinetic KSAW, where $\nu \to 0$

i.e "fluid" ↔ blob / fluid element of particles.

⇒ what holds blob together?
(i.e. prevents dispersal?)
⇒ collisions (i.e. particles collide and scatter prior dispersal)
or ⇒ vibrations in wave.

here focus on short mean-free path ordering.

For $C(f) \gg \partial f/\partial t$, $\underline{v}\cdot\nabla f$, etc.

1.O. $\quad C(f) = 0$

$$\Rightarrow f = f_{Maxwellian}$$

i.e. - collisions drive distribution function to local Maxwellian on time scale short compared all else
- n.b. Maxwellian can be shifted, and have gradients.

1st order:

$$\frac{\partial f^{(0)}}{\partial t} + \underline{v}\cdot\nabla f^{(0)} + \frac{q}{m}\left(\underline{E}+\frac{\underline{v}}{c}\times\underline{B}\right)\cdot\underline{\nabla} f^{(0)} = C(f^{(1)})$$

then integrating:

$$\int d^3v \left[\frac{\partial f^{(0)}}{\partial t} + \nabla\cdot v f^{(0)} + \overset{\text{IBP}\ 0}{\frac{\partial}{\partial v}\cdot\left(\frac{q}{m}\left(\underline{E}+\frac{\cancel{v}}{c}\times\underline{B}\right)f^{(0)}\right)}\right]$$

$$= \int d^3v\, C(f^{(1)})$$

Now, $\int d^3v\, C(f_i) = 0$ → collisions conserve #!

so, have:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\underline{V}) = 0$$

i.e. continuity equation!

$$\begin{cases} n = \int d^3v \ f \\ \underline{V} \equiv \int d^3v \ \underline{v} f \Big/ n \end{cases}$$ → basic moments.

→ Now first order moment:

$$\int d^3v\, \underline{v} \left(\overset{\textcircled{1}}{m\frac{\partial f}{\partial t}} + \overset{\textcircled{2}}{\underline{v} \cdot \underline{\nabla} f m} + \overset{\textcircled{3}}{q \left(\underline{E} + \underline{v} \times \underline{B} \right) \cdot \underline{\nabla}_v f} = \overset{\textcircled{4}}{C(f)} \right)$$

$$\textcircled{1} = m \frac{\partial}{\partial t}\left(n \underline{V}\right) \qquad \underline{V} = V(\underline{x}, t)$$

$$\textcircled{3} = \int \underline{v} q(\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial f}{\partial \underline{v}}$$

$$= \int \frac{\partial}{\partial \underline{v}} \left[f \underline{v} \cancelto{0}{(\underline{E} + \underline{v} \times \underline{B})} \right] d^3v - \int f \underline{v}\, \frac{\partial}{\partial \underline{v}} \cdot \left(\cancelto{0}{\underline{E}} + \cancelto{\perp \underline{v}}{\underline{v} \times \underline{B}} \right)$$

$$- \int f \left(\underline{E} + \underline{v} \times \underline{B} \right) \cdot \frac{\partial \underline{v}}{\partial \underline{v}}$$

<u>So</u>

$$= -qn\left(\underline{E} + \underline{V} \times \underline{B}\right)$$

$$\textcircled{4} = \int d^3v\, m C(f)\, \underline{v}$$

$$= \underline{P}_{ij}$$

→ collisional momentum transfer from species i to j which leaves ②;

$$② = m\int d^3v \ \underline{v}(\underline{v}\cdot\underline{\nabla})f$$

$$= m\int d^3v \ \underline{\nabla}\cdot(f\underline{v}\,\underline{v})$$

$$= \underline{\nabla}\cdot\left[m\int d^3v \, f\,\underline{v}\,\underline{v}\right] = m\,\underline{\nabla}\cdot(n\,\overline{\underline{v}\underline{v}})$$

clearly useful to separate $\underline{v}$ into mean and fluctuating pieces

$$\underline{v} = \underline{V} + \underline{w}$$

$$\Rightarrow \underline{\nabla}\cdot(n\,\overline{\underline{v}\underline{v}}) = \underline{\nabla}\cdot(n\underline{V}\,\underline{V}) + \underline{\nabla}\cdot(n\,\overline{\underline{w}\underline{w}})$$

$$+ \underbrace{\cancel{\underline{\nabla}\cdot n(\underline{V}\,\overline{\underline{w}} + \overline{\underline{w}}\,\underline{V})}}_{0,\ \text{defn.}}$$

$$\underline{\nabla}\cdot(n\underline{V}\,\underline{V}) = \underline{V}\ \underline{\nabla}\cdot(n\underline{V}) + n(\underline{V}\cdot\underline{\nabla})\underline{V}$$

$$n\,\overline{\underline{w}\underline{w}} \equiv \underline{\underline{P}}$$

↓

pressure tensor!

so, can write for momentum equation

$$m \frac{\partial}{\partial t}(n\underline{V}) + m\underline{V}\,\underline{\nabla}\cdot(n\underline{V}) + mn(\underline{V}\cdot\underline{\nabla})\underline{V}$$
$$+ \underline{\nabla}\cdot\underline{P} - qn(\underline{E} + \underline{V}\times\underline{B}) = R_{ij}$$

and using continuity:

$$\boxed{mn\left[\frac{\partial}{\partial t}\underline{V} + \underline{V}\cdot\underline{\nabla}\,\underline{V}\right] = qn(\underline{E} + \underline{V}\times\underline{B}) - \underline{\nabla}\cdot\underline{\underline{P}} + \underline{R_{ij}}}$$

Now, for form $\underline{P}$:

$$\underline{\underline{P}} = \int d^3v\; m\, v_i v_j f$$

in short mean-free-path ordering,

$$f \cong f_{Maxwellian}$$

As mean extracted, symmetry ⇒

$$\underline{\underline{P}} = \int d^3v\; v_i v_j \delta_{ij}\, f$$

so $$\underline{\underline{p}} = \begin{bmatrix} p_1 & & 0 \\ 0 & p_2 & \\ & & p_3 \end{bmatrix}$$ pressure tensor diagonal

if isotropic : $p_1 = p_2 = p_3$ (fast $\parallel, \perp$ thermal equilibration)

$\Rightarrow$
$$\underline{\underline{p}} = p \, \underline{\underline{\underline{I}}}$$

and pressure reduces to scalar, i.e.

$$mn\left[\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V}\right] = qn\left(\underline{E} + \underline{V}\times\underline{B}\right) - \underline{\nabla} p + \underline{P_{ij}}$$

$\rightarrow$ For second order moment $\rightarrow$ energy (closure $\Leftrightarrow$ energy flux! $_0$) $\Rightarrow$ eqn. state

2 species $\Rightarrow$ p/ρ^{γ} = const.

→ Single Fluid (→ MHD)

Can define single fluid variables:

$$\rho = n_i M + n_e M \cong n M$$ → density

mass velocity:

mean velocity

$$\underline{V} = \frac{1}{\rho}\left(n_i M \underline{V}_i + n_e M_e \underline{V}_e\right)$$

$$\approx \left[\frac{M \underline{V}_i + m_e \underline{V}_e}{M + m}\right] \cong \underline{V}_i$$

Current density:

relative velocity

$$\underline{J} = q\,(n_i \underline{V}_i - n_e \underline{V}_e)$$

$$\approx nq\,(\underline{V}_i - \underline{V}_e)$$, using QN.

Upshot:

- Continuity for ions ⇒ single fluid continuity

$$\therefore \quad \boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{V}) = 0}$$

- adding electron and ion momentum eqns:

$$Mn\left(\frac{\partial \underline{V}_i}{\partial t} + \underline{V}_i\cdot\underline{\nabla}\,\underline{V}_i\right) = qn\left(\underline{E} + \underline{V}_i\times\underline{B}\right) - \underline{\nabla}p_i + \underline{P}_{i,e}$$

$$\oplus$$

$$m_e n\left(\frac{\partial \underline{V}_e}{\partial t} + \underline{V}_e\cdot\underline{\nabla}\,\underline{V}_e\right) = -qn\left(\underline{E} + \underline{V}_e\times\underline{B}\right) - \underline{\nabla}p_e + \underline{P}_{e,i}$$

$\Rightarrow$

$$n\left(\frac{\partial}{\partial t}\left(M\underline{V}_i + m\underline{V}_e\right) + M(\underline{V}_i\cdot\underline{\nabla})\underline{V}_i + m_e(\underline{V}_e\cdot\underline{\nabla})\underline{V}_e\right) = qn\left(\underline{V}_i - \underline{V}_e\right)\times\underline{B} - \underline{\nabla}(p_i + p_e) \underset{\substack{\underline{0} \\ \text{momentum cons.}}}{\cancel{+\underline{P}_{e,i} + \underline{P}_{i,e}}}$$

as: $m_e \ll M$

$\underline{J}$ defn.

$\rho = \rho_e + \rho_i$

$$\Rightarrow \boxed{\rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V}\right) = \underline{J}\times\underline{B} - \underline{\nabla} p + \underset{\substack{\downarrow \\ \text{any additional} \\ \text{body force.}}}{F_{body}}}$$

Momentum balance.

⇒ Now, only re-maining <u>non</u>-<u>trivial</u> MHD equation is Ohm's Law.

→ where the bodies are buried....

Consider, $\left[m_e * (\text{ion momentum eqn}) - M * (\text{electron momentum eqn.})\right]$

$$\Rightarrow M m_e n\left(\frac{\partial}{\partial t}(\underline{V}_i - \underline{V}_e) + \underline{V}_i\cdot\underline{\nabla}\,V_i - \underline{V}_e\cdot\underline{\nabla}\,\underline{V}_e\right)$$

$$= qn\,(M + m_e)\,\underline{E} + qn\,(m\underline{V}_i + M\underline{V}_e)\times\underline{B}$$

$$- m\nabla P_i + M\nabla P_e - (M+m)\,\underline{P}_{ei}$$

Now, ① $\underline{P}_{ei}$ = electron-ion momentum transfer

$$= -Mnq\eta\underline{J}$$

② $M >> m_e$

③ neglecting advective derivatives

$\Rightarrow$

$$-\frac{M m_e n}{q}\frac{\partial}{\partial t}\left(\frac{\underline{J}}{n}\right) = q\rho\underline{E} - Mnq\eta\underline{J} + M\nabla P_e + qn(m\underline{V}_i + M\underline{V}_e)\times\underline{B}$$

and can further simplify:

$$m\underline{V}_i + M\underline{V}_e = M\underline{V}_i + m\underline{V}_e - (M-m)(\underline{V}_i - \underline{V}_e) \cong \frac{\rho\underline{V}}{n} - \frac{M}{nq}\underline{J}$$

Finally, re-arrangeing $\Rightarrow$

$$\boxed{\begin{aligned}\frac{m_e}{nq^2}\frac{\partial\underline{J}}{\partial t} = \left(\underline{E} + \frac{\underline{V}\times\underline{B}}{c}\right) - \eta\underline{J} - \frac{1}{nq}(\underline{J}\times\underline{B}) \\ + \frac{1}{nq}\nabla P_e\end{aligned}}$$

Now, have generalized Ohm's Law:

$$\overset{①}{\frac{m_e}{nq^2}\frac{\partial \underline{J}}{\partial t}} = \overset{②}{\left(\underline{E} + \frac{\underline{V}\times\underline{B}}{\text{⑤}\,c}\right)} - \overset{③}{\eta\underline{J}} - \overset{④}{\left(\frac{\underline{J}\times\underline{B}}{nq}\right)} + \frac{\nabla p_e}{nq}$$

② → ideal MHD Ohm's Law

③ → collisional resistivity

} resistive MHD

'ring on ④ : Hall Term

⇒ Hall MHD

bring in ⑤ : Electron thermal force / pressure

⇒ diamagnetic / finite electron ω_* MHD

i.e. Boltzmann response : $\underline{E}$ vs $\frac{\nabla p_e}{nq}$

① : Electron inertia term ($\sim m_e$)

⇒ EMHD, electron inertially modified MHD.

($\omega\, m_e/nq^2 > \eta$)

For low frequency, strong collisionality, etc.

$$\Rightarrow \quad \underline{E} + \underline{V}\times\underline{B} = \overset{\nearrow}{\eta}\,\underline{J} \quad \Bigg\} \; \text{Resistive MHD.}$$

N.B. :- Ohm's Law is most sensitive part of MHD structure → need care.

- high ω → electron inertia
  tok. μ-inst → thermal force term.
  $\lambda \sim c/\omega_{pi}^2$ → Hall term.